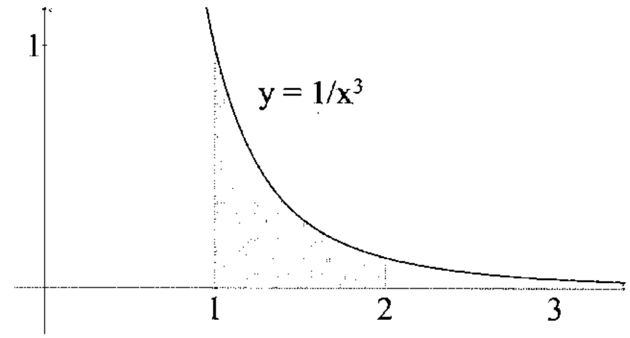
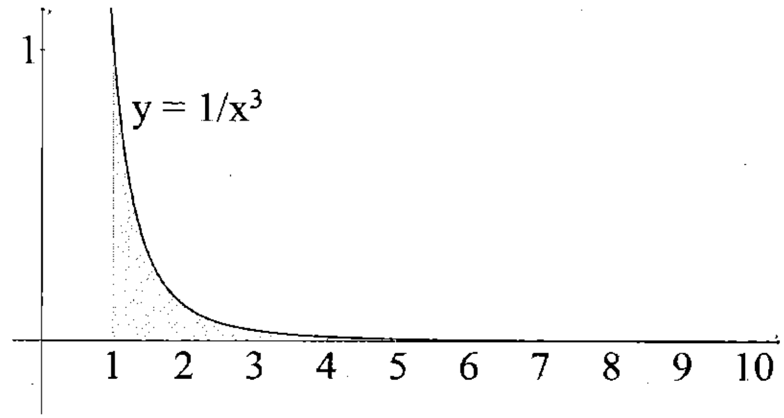




7.8 Improper Integrals

Motivation: Consider the function $f(x) = \frac{1}{x^3}$. Compute the area under the function from…

1. x = 1 to x = t
2. x = 1 to x = 10
3. x = 1 to x = 100

$$\int_1^t \frac{1}{x^3}\,dx = \int_1^t x^{-3}\,dx = \frac{1}{-2}x^{-2}\Big|_1^t$$

$$= -\frac{1}{2x^2}\Big|_1^t$$

$$= -\frac{1}{2t^2} - -\frac{1}{2} = -\frac{1}{2t^2} + \frac{1}{2}$$

$$\int_1^{10} \frac{1}{x^3}\,dx = -\frac{1}{2}\,\frac{1}{10^2} + \frac{1}{2} = 0.495$$

$$\int_1^{100} \frac{1}{x^3}\,dx = -\frac{1}{2}\,\frac{1}{100^2} + \frac{1}{2} = 0.49995$$

Def'n: *Improper type 1 - infinite integral of integration*

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx$$

If the limit exists and is finite, then we say the integral *converges*.
Otherwise, we say it *diverges*.

Example:

$$\int_1^\infty \frac{1}{x^3}dx = \lim_{t\to\infty}\left[\int_1^t x^{-3}dx\right]$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2}x^{-2}\Big|_1^t\right]$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2x^2}\Big|_1^t\right]$$

$$= \lim_{t\to\infty}\left[\underbrace{-\frac{1}{2t^2}}_{0} - -\frac{1}{2}\right]$$

$$= 0 + \frac{1}{2}$$

$$= \boxed{1/2}$$

CONVERGES!

Example:

$$\int_{-1}^{\infty} e^{-2x}\,dx = \lim_{t\to\infty} \int_{-1}^{t} e^{-2x}\,dx$$

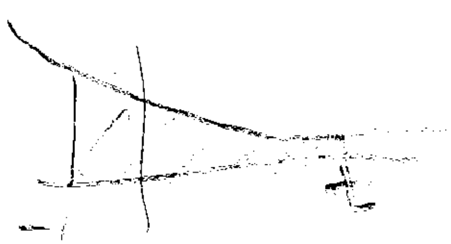
$$= \lim_{t\to\infty} \left[-\frac{1}{2} e^{-2x} \Big|_{-1}^{t} \right]$$

$$= \lim_{t\to\infty} \left[-\frac{1}{2} e^{-2t} - -\frac{1}{2} e^{2} \right]$$

$$= \lim_{t\to\infty} \left[\underbrace{-\frac{1}{2} e^{-2t}}_{\text{SAME AS } -\frac{1}{2e^{2t}}} + \frac{1}{2} e^{2} \right]$$

$$= 0 + \frac{1}{2} e^{2} = \boxed{\frac{1}{2} e^{2}}$$

$\boxed{\text{CONVERGES}}$



Example:

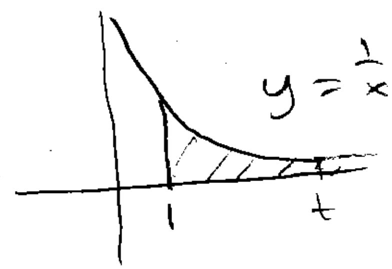
$$\int_{1}^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \left[\int_{1}^{t} \frac{1}{x}\,dx \right]$$

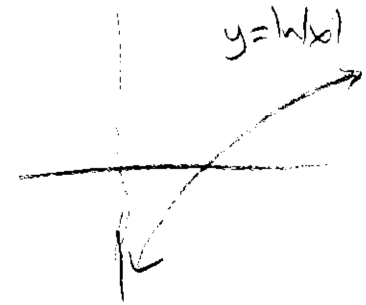
$$= \lim_{t\to\infty} \left[\ln|x| \Big|_{1}^{t} \right]$$

$$= \lim_{t\to\infty} \left[\ln(t) - \underbrace{\ln(1)}_{0} \right]$$

$$= \lim_{t\to\infty} \ln(t) = \infty$$

$\boxed{\text{DIVERGES}}$

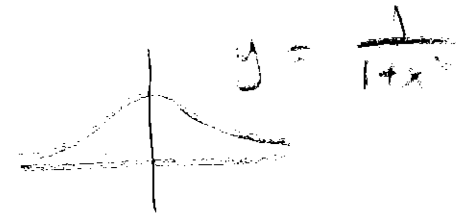




Def'n:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{r\to-\infty}\int_{r}^{0} f(x)dx + \lim_{t\to\infty}\int_{0}^{t} f(x)dx$$

In this case, we say it *converges* only if both limits <u>separately</u> exist and are finite.



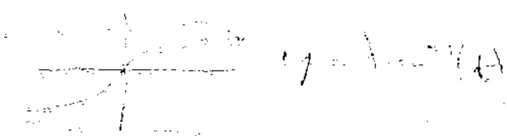
Example:

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \int_{-\infty}^{0}\frac{1}{1+x^2}dx + \int_{0}^{\infty}\frac{1}{1+x^2}dx$$

$$= \lim_{r\to-\infty}\left[\int_{r}^{0}\frac{1}{1+x^2}dx\right] + \lim_{t\to\infty}\left[\int_{0}^{t}\frac{1}{1+x^2}dx\right]$$

$$= \lim_{r\to-\infty}\left[\underbrace{\tan^{-1}(0)}_{0} - \tan^{-1}(r)\right] + \lim_{t\to\infty}\left[\tan^{-1}(t) - \underbrace{\tan^{-1}(0)}_{0}\right]$$

$$= -(-\pi/2) + \pi/2 = \boxed{\pi}$$



Def'n: *Improper type 2 - infinite discontinuity*

If $f(x)$ has a discontinuity at x = a, then

$$\int_a^b f(x)dx = \lim_{t\to a^+} \int_t^b f(x)dx$$

If $f(x)$ has a discontinuity at x = b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

If the limit exists and is finite, then we say the integral *converges*.

Otherwise, we say it *diverges*.

Example:

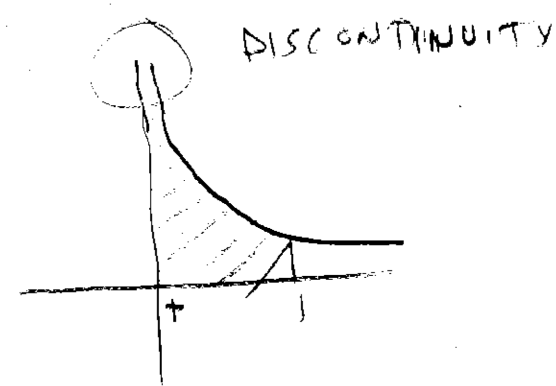
$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{t\to 0^+}\left[\int_t^1 x^{-1/2}dx\right]$$

$$= \lim_{t\to 0^+}\left[2x^{1/2}\Big|_t^1\right]$$

$$= \lim_{t\to 0^+}\left[2\sqrt{1} - 2\sqrt{t}\right]$$

$$= 2 - 0 = \boxed{2}$$

CONVERGES



Example:

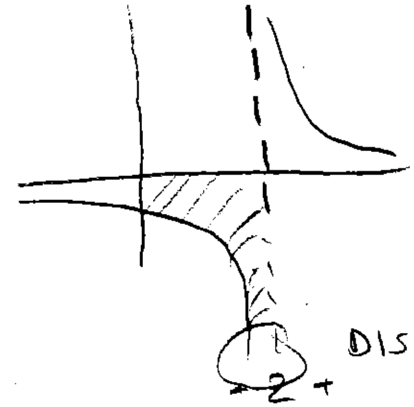
$$\int_0^2 \frac{x}{x-2}\,dx = \lim_{t\to 2^-}\left[\int_0^t \frac{x}{x-2}\,dx\right]$$

$$= \lim_{t\to 2^-}\left[\int_0^t 1 + \frac{2}{x-2}\,dx\right]$$

$$= \lim_{t\to 2^-}\left[\, x + 2\ln|x-2| \,\Big|_0^t\,\right]$$

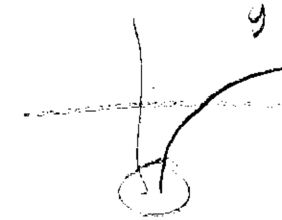
$$= \lim_{t\to 2^-}\left[\left(\cancel{t^2} + \underbrace{2\ln|\underbrace{t-2}_{0}|}_{-\infty}\right) - \left(0 + 2\ln(2)\right)\right]$$

$\boxed{\text{DIVERGES}}$



DISCONTINUITY

$$\begin{array}{r} 1 \\ x-2\,\overline{)\,x} \\ -(x-2) \\ \hline 2 \end{array}$$



$y = \ln(x)$

If $f(x)$ has a discontinuity at $x = c$ which is **between** a and b, then

$$\int_a^b f(x)dx = \lim_{r\to c^-}\int_a^r f(x)dx + \lim_{t\to c^+}\int_t^b f(x)dx$$

In this case, we say it *converges* only if both limits separately exist and are finite.

Example:

$$\int_0^\pi \frac{1}{\cos^2(x)}dx = \int_0^\pi \sec^2(x)dx$$

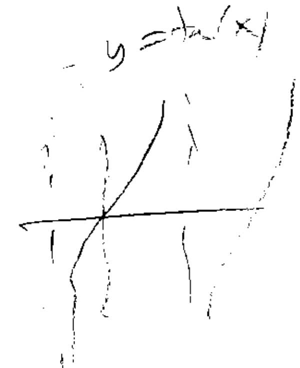
$\cos(x) = 0$ AT $x = \pi/2$

$$= \lim_{r\to \pi/2^-}\left[\int_0^r \sec^2(x)dx\right] + \lim_{t\to \pi/2^+}\left[\int_t^\pi \sec^2(x)dx\right]$$

$$= \lim_{r\to \pi/2^-}\left(\tan(x)\Big|_0^r\right) + \lim_{t\to \pi/2^+}\left[\tan(x)\Big|_t^\pi\right]$$

$$= \lim_{r\to \pi/2^-}[\underbrace{\tan(r)}_{+\infty} - 0] + \lim_{t\to \pi/2^+}[\underbrace{0 - \tan(t)}_{+\infty}]$$

STOP



$\boxed{\text{DIVERGES}}$

Limits Refresher

1. If stuck, plug in values "near" t.
2. Know your basic functions/values:

$$\lim_{t\to\infty} \frac{1}{t^a} = 0\,, \qquad \text{if } a > 0.$$

$$\lim_{t\to\infty} \frac{1}{e^{at}} = 0\,, \qquad \text{if } a > 0.$$

$$\lim_{t\to\infty} t^a = \infty\,, \qquad \text{if } a > 0.$$

$$\lim_{t\to\infty} \ln(t) = \infty.$$

$$\lim_{t\to 0^+} \ln(t) = -\infty.$$

3. For indeterminant forms, use algebra and/or L'Hopital's rule

Examples:

$$\lim_{t\to 1} \frac{t^2+2t-3}{t-1} \overset{\frac{0}{0}}{=} \lim_{t\to 1} \frac{(t-1)(t+3)}{(t-1)} = 4$$

$$\lim_{t\to\infty} \frac{\ln(t)}{t} \overset{\frac{\infty}{\infty}}{=} \lim_{t\to\infty} \frac{1/t}{1} = 0$$

$$\lim_{t\to\infty} t^2 e^{-3t} = \lim_{t\to\infty} \frac{t^2}{e^{3t}} \overset{\frac{\infty}{\infty}}{=} \lim_{t\to\infty} \frac{2t}{3e^{3t}} \overset{\frac{\infty}{\infty}}{=} \lim_{t\to\infty} \frac{2}{9e^{3t}} = 0$$

Example: (LIKE HW)

$$\int_0^\infty xe^{-x^2}\,dx$$

$$= \lim_{t\to\infty}\left[\int_0^t xe^{-x^2}\,dx\right]$$

$u = -x^2$

$du = -2x\,dx$

$-\frac{1}{2x}du = dx$

$$= \lim_{t\to\infty}\left[\int_0^{-t^2} x\,e^{u}\frac{1}{-2x}\,du\right]$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2}e^{u}\Big|_0^{-t^2}\right]$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2}e^{-t^2} - -\frac{1}{2}e^{0}\right]$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2e^{t^2}} + \frac{1}{2}\right]$$

$$= \boxed{\frac{1}{2}}$$

$\boxed{\text{CONVERGES}}$

Example:

$$\int_0^\infty xe^{-2x}dx$$

$$\lim_{t\to\infty}\left[\int_0^t xe^{-2x}dx\right]$$

$u = x \quad dv = e^{-2x}dx$

$du = dx \quad v = -\frac{1}{2}e^{-2x}$

$$\lim_{t\to\infty}\left[-\frac{1}{2}xe^{-2x}\Big|_0^t - \int_0^t -\frac{1}{2}e^{-2x}dx\right]$$

$$= \lim_{t\to\infty}\left[\left(-\frac{1}{2}te^{-2t} - 0\right) - \frac{1}{4}e^{-2x}\Big|_0^t\right]$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2}\frac{t}{e^{2t}} - \left(\frac{1}{2}e^{-2t} - \frac{1}{4}\right)\right]$$

$$= \lim_{t\to\infty}\left[-\frac{t}{2e^{2t}} - \frac{1}{2e^{2t}}^{\to 0} + \frac{1}{4}\right] = \boxed{\frac{1}{4}}$$

$\frac{t}{2e^{2t}} \to \frac{\infty}{\infty}$

L'Hopital's

$\frac{1}{4e^{2t}} \to 0$

Converges

Aside:
A few general notes on **comparison**:

Suppose you have two functions $f(x)$ and $g(x)$ such that

$$0 \le g(x) \le f(x)$$

for all values of x.

(a) If $\int_a^\infty f(x)dx$ converges, then $\int_a^\infty g(x)dx$ converges.

(b) If $\int_a^\infty g(x)dx$ diverges, then $\int_a^\infty f(x)dx$ diverges.

You can verify that

$$\int_1^\infty \frac{1}{x^p}\,dx\,, \qquad \text{converges for } p > 1.$$

$$\int_1^\infty e^{px}\,dx\,, \qquad \text{converges for } p < 0.$$

You can compare off of these to sometimes quickly tell if an integral converges/diverges (without computing)

Example:

$$\int_1^\infty \frac{1}{x^4 + x}\,dx \text{ converges}$$

because

1. $\frac{1}{x^4+x} < \frac{1}{x^4}$ for all $x > 1$, and
2. $\int_1^\infty \frac{1}{x^4}\,dx$ converges.

Example:

$$\int_1^\infty \frac{2 + \cos(x)}{x}\,dx \text{ diverges}$$

because

1. $\frac{2+\cos(x)}{x} > \frac{1}{x}$ for all $x > 1$, and
2. $\int_1^\infty \frac{1}{x}\,dx$ diverges.